



## ANALYSIS OF LINEAR DIFFERENTIAL SYSTEMS WITH ARROWHEAD MATRIX STRUCTURE

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### ABSTRACT

*The paper will provide a systematic analysis, which is rigorous, of linear differential systems under arrowhead matrix structures. The centralized coupling and structural constraints of such matrices occur naturally in a general range of scientific or engineering applications. It involves linear systems theory, spectral analysis, and matrix theory tools as part of the study to examine the qualitative behavior of such systems. Some of the important findings are explicit characterization of spectral properties, eigenvalue localization, and stability conditions based on Gershgorin-based arguments as well as on Lyapunov arguments. This hierarchical nature of the arrowhead-matrices is taken advantage of to have a representation of the system using components, a natural separation between the dominating and sub-dominating subsystems, and a framework of modal solutions. Examples are given with tabular validation to show how conclusions on stability can be made based on the parameters of the matrix without necessarily computing the eigenvalues. The findings point to the analytical tractability and the practical applicability of the arrowhead matrices as a unifying framework to the study of the structured linear dynamical systems.*

**Keywords:** *Linear differential systems, Arrowhead matrices, Spectral analysis, Stability analysis, Lyapunov methods, Structured matrices, Dynamical systems.*

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### 1. INTRODUCTION

Finite-dimensional linear differential systems constitute a fundamental framework for modeling and analyzing dynamical processes across a wide range of scientific and engineering disciplines. Classical linear systems theory establishes that the qualitative behavior of such systems—including stability, convergence, and long-term evolution—is primarily governed by the spectral properties of the system matrix, such as eigenvalue distributions, modal structures, and invariant subspaces. These elements provide deep insight into how system trajectories evolve over time and form the theoretical backbone for analyzing linear dynamical behavior.

However, in practical applications, system matrices are rarely arbitrary; instead, they often exhibit specific algebraic structures arising from physical constraints, modeling assumptions, or numerical discretization techniques. The presence of such structure significantly enhances both analytical tractability and computational efficiency, as it allows researchers to exploit additional mathematical properties that lead to sharper stability criteria, reduced-order models, and more efficient numerical algorithms.

Among structured matrices, arrowhead matrices hold particular importance due to their distinctive configuration, characterized by a dominant row and column coupled with an otherwise diagonal or block-diagonal structure. These matrices naturally arise in various real-world contexts, including vibration systems with a dominant mass, centralized feedback control mechanisms, ecological interaction networks with a key species or resource, and sparse matrix problems in numerical linear algebra. The rank-one coupling inherent in arrowhead matrices introduces nontrivial dynamical interactions while still preserving a high degree of analytical simplicity, making them especially attractive for theoretical investigation. Motivated by these properties, the present study undertakes a systematic and rigorous analysis of linear differential systems governed by arrowhead-structured matrices.

By integrating tools from linear systems theory, spectral analysis, and structured matrix theory, the paper establishes stability conditions, characterizes eigenvalue behavior, and derives explicit solution representations. This analysis not only clarifies the role of structural coupling in shaping system dynamics but also demonstrates the effectiveness of arrowhead matrices as a unifying framework for studying structured linear dynamical systems with both theoretical depth and practical relevance.

## 2. LITERATURE REVIEW

**Alves et al. (2019)** investigated the identification of linear absolute instabilities in fluid systems through differential eigenvalue problems, employing sensitivity analysis as a key analytical tool. The study developed a structured methodology to examine how small variations in system parameters influenced the stability and transition behavior of fluid flows. It demonstrated that sensitivity analysis enabled precise identification of critical parameters responsible for instability onset, thereby improving predictive accuracy. The research provided valuable insights into instability mechanisms in fluid mechanics and contributed to the development of advanced analytical techniques for studying complex dynamical systems.

**Belov et al. (2017)** proposed the arrowhead decomposition method for efficiently solving block-tridiagonal systems of linear equations, which frequently arise in large-scale computational models. The study showed

that this method transformed complex matrix systems into simpler structures, reducing computational burden and improving numerical stability. It emphasized that the decomposition approach allowed faster solution procedures while maintaining high accuracy, making it particularly useful in simulations involving large datasets and structured matrices in physics and engineering applications.

**Bezanson et al. (2017)** introduced the Julia programming language as a high-performance computing environment designed specifically for numerical and scientific computing. The study highlighted Julia's unique ability to combine user-friendly syntax with execution speeds comparable to low-level languages such as C and Fortran. It demonstrated that Julia supported advanced features such as parallel computing, just-in-time compilation, and efficient handling of large-scale numerical problems. The research emphasized that Julia significantly enhanced productivity for researchers and engineers by enabling rapid prototyping without compromising computational efficiency.

**Bullo and Lewis (2019)** explored the geometric control of mechanical systems by presenting a comprehensive framework that integrated differential geometry with control theory. The study focused on modeling, analysis, and design of mechanical systems using geometric principles, enabling a deeper understanding of system dynamics and structural properties. It demonstrated that geometric control methods provided powerful tools for analyzing stability, controllability, and motion planning in complex systems. The work was particularly relevant to applications in robotics, aerospace engineering, and mechanical systems design, where precise control and stability are essential.

### 3. PRELIMINARIES AND ARROWHEAD MATRIX STRUCTURE

This part presents the basic mathematical terms and notation needed in the further analysis. Basic results on linear autonomous differential systems and their representation of solutions, the basis of analytical foundations of the study, are recalled first. The arrowhead matrix structure is then defined and its main properties discussed that would be central to the spectral and stability analysis that is developed in subsequent sections.

#### 3.1 Linear Differential Systems

Consider the autonomous linear differential system

$$\dot{x}(t) = Ax(t), x(0) = x_0,$$

where  $x(t) \in \mathbb{R}^n$  denotes the state vector and  $A \in \mathbb{R}^{n \times n}$  is a constant system matrix. Such systems form the basic class of continuous-time linear dynamical models and arise frequently in control theory, physics, and applied mathematics.

For any initial condition  $x_0 \in \mathbb{R}^n$ , the system admits a unique solution defined for all  $t \geq 0$ , which can be expressed in closed form as

$$x(t) = e^{At}x_0,$$

where  $e^{At}$  denotes the matrix exponential. The qualitative behavior of the solution, its stability, and its behavior over time are determined in terms of the spectral properties of matrix  $A$ , such as its eigenvalues and invariant subspaces.

### 3.2 Arrowhead Matrices

#### Definition 3.1

A matrix  $A \in \mathbb{R}^{n \times n}$  is called an **arrowhead matrix** if it has the form

$$A = \begin{pmatrix} \alpha & b_1 & b_2 & \cdots & b_{n-1} \\ c_1 & d_1 & 0 & \cdots & 0 \\ c_2 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & d_{n-1} \end{pmatrix},$$

where  $\alpha, d_i \in \mathbb{R}$  and  $b_i, c_i \in \mathbb{R}$  for  $i = 1, \dots, n-1$ . In this structure, the first row and first column contain the coupling terms, while the remaining submatrix is diagonal.

If  $b_i = c_i$  for all  $i$ , the arrowhead matrix is symmetric. Symmetric arrowhead matrices frequently arise in mechanical and physical systems and possess real eigenvalues, which further simplifies their spectral analysis.

#### Remark 3.1

The Arrowhead matrices can also be viewed as diagonal matrices with a rank-one perturbation to add coupling as a result of a single dominant state variable. The spectral and stability properties of the system are also highly informative using this interpretation, since the major behavior of the system is highly

controlled by the diagonal entries, whereas the coupling terms only affect a small fraction of the spectrum. Thus, arrowhead matrices provide a good trade-off between structure simplicity and dynamical richness and are therefore especially susceptible to analytical studies.

#### 4. SPECTRAL PROPERTIES OF ARROWHEAD MATRICES

In fact, the spectral characteristics of arrowhead matrices play a very decisive role in qualitatively characterizing the behavior of associated linear differential systems. Because of their structured form, explicit analytical expressions of the characteristic polynomials of arrowhead matrices are available that possess typical patterns of eigenvalue distributions. Such properties allow precise eigenvalue localization and facilitate stability analysis.

##### Theorem 4.1 (Characteristic Polynomial)

Let  $A \in \mathbb{R}^{n \times n}$  be an arrowhead matrix as defined in Section 3. The characteristic polynomial of  $A$  satisfies

$$\det(A - \lambda I) = (\alpha - \lambda) \prod_{i=1}^{n-1} (d_i - \lambda) - \sum_{i=1}^{n-1} b_i c_i \prod_{j \neq i} (d_j - \lambda).$$

##### Proof

The result follows by applying a block determinant expansion along the first row and first column of  $A - \lambda I$ . The arrowhead structure makes sure that every off-diagonal interaction is restricted within a row and a column which then gives the determinant as a product of diagonal interactions and a product of terms of coupling.

##### Theorem 4.2 (Eigenvalue Localization)

For an arrowhead matrix  $A$ , all but at most one eigenvalue are closely associated with the diagonal entries  $d_i$ . The remaining eigenvalue arises due to the rank-one coupling induced by the first row and column and is primarily governed by the magnitude of the coupling coefficients  $b_i$  and  $c_i$ .

##### Remark 4.1

The fact that eigenvalues can be separated into the ones that are related to diagonal elements and those that are affected by the coupling is one of the primary characteristics of arrowhead matrices. This spectral

structure makes localization of eigenvalues useful and is useful in stability analysis, in reduced-order modeling and hierarchical system representations. Specifically, weak coupling results in the low levels of deviation off the diagonal spectrum, but stronger coupling may cause large changes in the dominant eigenvalue.

## 5. STABILITY AND LYAPUNOV ANALYSIS

Consider the linear autonomous system

$$\dot{x}(t) = Ax(t),$$

where  $A \in \mathbb{R}^{n \times n}$  is an arrowhead matrix as defined in Section 3. The equilibrium point  $x=0$  is stable depending on the spectral properties of  $A$  and also on the interaction between the diagonal dynamics and the coupling terms. Because of the particular form of the matrices of arrows, the spectral as well as the Lyapunov-based conditions of stability can be stated explicitly and in a form that is amenable to an analysis.

### Theorem 5.1 (Gershgorin Stability Criterion)

Suppose that

$$d_i < 0 \text{ for all } i,$$

and

$$\alpha + \sum_{i=1}^{n-1} \frac{|b_i c_i|}{|d_i|} < 0.$$

Then the equilibrium  $x = 0$  of the arrowhead system is asymptotically stable.

**Proof**

Both Gershgorin circle and circle theorem imply that any eigenvalue of  $A$  is contained in at least one Gershgorin disc centered at a diagonal entry with radius equal to the sum of the absolute values of the corresponding off-diagonal entries. Due to the strictly negative diagonal elements, and the perturbation introduced by coupling satisfying the mentioned inequality, all the Gershgorin discs fall within the open left half-plane. As a result, the real part of all the eigenvalues of  $A$  is strictly negative, which means that the equilibrium is asymptotically stable.

### Theorem 5.2 (Lyapunov Stability Criterion)

If there exists a diagonal positive definite matrix  $P$  such that

$$A^T P + P A < 0,$$

then the equilibrium  $x = 0$  of the arrowhead system is globally asymptotically stable.

### Proof

Consider the quadratic Lyapunov function  $V(x) = x^T P x$ , which is positive definite due to the positivity of  $P$ . The time derivative of  $V$  along system trajectories satisfy

$$\dot{V}(x) = x^T (A^T P + P A) x.$$

The negative definiteness of  $A^T P + P A$  implies  $\dot{V}(x) < 0$  for all  $x \neq 0$ , thereby establishing global asymptotic stability of the equilibrium.

### Remark 5.1

Lyapunov-based stability verification is made very easy by the arrowhead structure of the system matrix. Specifically, with  $P$  is selected as a diagonal, the matrix inequality  $A^T P + P A < 0$  simplifies to a system of scalar inequalities of the diagonal and coupling terms. This simplifies the task of stability analysis, and offers an analytically transparent method of analysis that is computationally efficient, particularly in high-dimensional systems that are centralizedly coupled.

## 6. STRUCTURAL DECOMPOSITION AND SOLUTION REPRESENTATION

The natural hierarchical architecture of the system dynamics is caused by the arrowhead structure of the system matrix. This section uses this structure to obtain a componentwise representation of the system, a decomposition result and give the resultant solution form. These properties play a central role in explaining the dynamic interplay between the dominant state and the other subsystems.

### 6.1 Componentwise Representation

Consider the linear system

$$\dot{x}(t) = Ax(t),$$

Where A is an arrowhead matrix. Explicitly writing the system in componentwise form is the result of which is

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + \sum_{i=2}^n b_{i-1} x_i, \\ \dot{x}_i &= d_{i-1} x_i + c_{i-1} x_1, i = 2, \dots, n.\end{aligned}$$

This representation indicates that the first state variable  $x_1$  is a central or dominating component and it interacts with the rest of the state variables, where the rest of the state variables are conditionally decoupled (i.e., it does not interact) with each other.

### 6.2 Hierarchical Decomposition

#### Theorem 6.1 (Hierarchical Decomposition)

A linear differential system with its arrowhead matrix is firstly decomposed into a central subsystem dominated by a single central subsystem and secondly coupled with n-1 subordinate scalar subsystems.

#### Proof

From the componentwise equations, it is evident that each subsystem  $x_i$ ,  $i \geq 2$ , evolves independently of the other subordinate states and is influenced only by the central state  $x_1$ . The development of  $x_1$ , on the other hand, is based on the entire subordinate states. This creates a hierarchical structure of coupling, whereby a single dominant subsystem interacts with a number of lower-dimensional subsystems.

### 6.3 Modal Solution Representation

When the system matrix has an entirety of eigenvectors, the solution may be written in modal form.

#### Theorem 6.2 (Modal Solution Representation)

If the arrowhead matrix  $A$  is diagonalizable with eigenvalues  $\lambda_k$ , right eigenvectors  $v_k$ , and corresponding left eigenvectors  $w_k$ , then the solution of the system is given by

$$x(t) = \sum_{k=1}^n e^{\lambda_k t} v_k \langle w_k, x_0 \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product.

#### Proof

Since  $A$  is diagonalizable, it admits a spectral decomposition. The result follows directly from the standard eigen-expansion of the matrix exponential  $e^{At}$ .

#### Remark 6.1

The hierarchical breakup that the arrowhead structure imposes comes in specifically useful to reduced-order models, observer design, as well as hierarchical control policies. The key subsystem takes into account the dominating dynamics, whereas the subordinate subsystems may be well analyzed or simplified, which result in considerable simplifications in the analysis of the theory and its implementation.

## 7. ILLUSTRATIVE EXAMPLES AND APPLICATIONS

The section gives an illustrative example to show the theoretical findings made in the previous sections. The example illustrates the application of the spectral and stability conditions obtained in the case of arrowhead matrices in practice. Short comments on the areas of common use are also given to show wider applicability of analysis.

### Example 7.1

Consider the arrowhead matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -4 \end{pmatrix}.$$

The characteristic arrowhead structure can be observed in the matrix where the first row and the column have a way of introducing coupling between the central state and the rest of subsystems, and the submatrix in the bottom right corner is diagonal.  $A$  is strictly negative on all its diagonals, which is enough to ensure that the uncoupled subsystems are stable.

Table 1 sums up the numerical values of the system parameters, which depict this arrowhead structure.

**Table 1:** Parameters of the arrowhead system

| Parameter | Value | Description                |
|-----------|-------|----------------------------|
| $\alpha$  | -2    | Central diagonal entry     |
| $d_1$     | -3    | Subsystem 1 diagonal entry |
| $d_2$     | -4    | Subsystem 2 diagonal entry |
| $b_1$     | 1     | Coupling coefficient       |
| $b_2$     | 1     | Coupling coefficient       |
| $c_1$     | 1     | Coupling coefficient       |
| $c_2$     | 1     | Coupling coefficient       |

In addition, the values of the coupling coefficients are small in comparison with the diagonal terms, so that the stability criterion in terms of Gershgorin that is provided in Section 5 is fulfilled. This in turn implies that all the eigenvalues of  $A$  lie in the open left half-plane, and the equilibrium of the corresponding linear differential system,  $x=0$ , is asymptotically stable.

Result of verification of the stability conditions worked out in Section 5 of this example is presented in Table 2.

**Table 2.** Verification of stability conditions for

| Condition           | Requirement               | Result                |
|---------------------|---------------------------|-----------------------|
| Diagonal negativity | $d_i < 0$                 | Satisfied             |
| Coupling dominance  | $(\alpha + \sum b_i c_i)$ |                       |
| Eigenvalue location | $\text{Re}(\lambda) < 0$  | Confirmed             |
| Stability type      | —                         | Asymptotically stable |

This example illustrates how arrowhead structure allows stability conclusions to be drawn directly from matrix entries without explicit eigenvalue computation.

## 8. CONCLUSION

This paper has presented a detailed mathematical analysis of linear differential systems which have an arrowhead matrix structure with a particular focus on the relationship between the matrix structure, spectral behavior, and the stability of the system. Through the rank-one coupling of the arrowhead matrices, explicit formulae were obtained of both characteristic polynomials and eigenvalue localization which gave easily tractable stability conditions in both spectral and Lyapunov choices. The hierarchical reduction of the system dynamics indicated a distinct separation of dominant and subordinate subsystems, which allowed simplified analysis and reduced-order interpretations. The theoretical results were demonstrated to be easily applicable in practice with the help of an illustrative numerical example and supportive tables. Generally, the discussion has shown that the arrowhead matrices provide an effective trade-off between the simplicity of the structure and the complexity of the dynamics and are therefore best suited to the modelling, analysis, and control of structured linear dynamical systems.

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